Kirchhoff-Helmholtz Integral Theorem

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \vec{f} - \nabla p'$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v} = q$$

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p' = -\frac{\partial q}{\partial t} + \nabla \cdot \vec{f}$$

Time harmonic case: $p' = \tilde{p}\, e^{-i\omega t}$

$$(\nabla^2 + k^2)\,\tilde{p} = i\omega \tilde{q} + \nabla \cdot \vec{\tilde{f}}$$

$k = \omega / c_0$, $\vec{f} = \vec{\tilde{f}}\, e^{-i\omega t}$, $q = \tilde{q}\, e^{-i\omega t}$

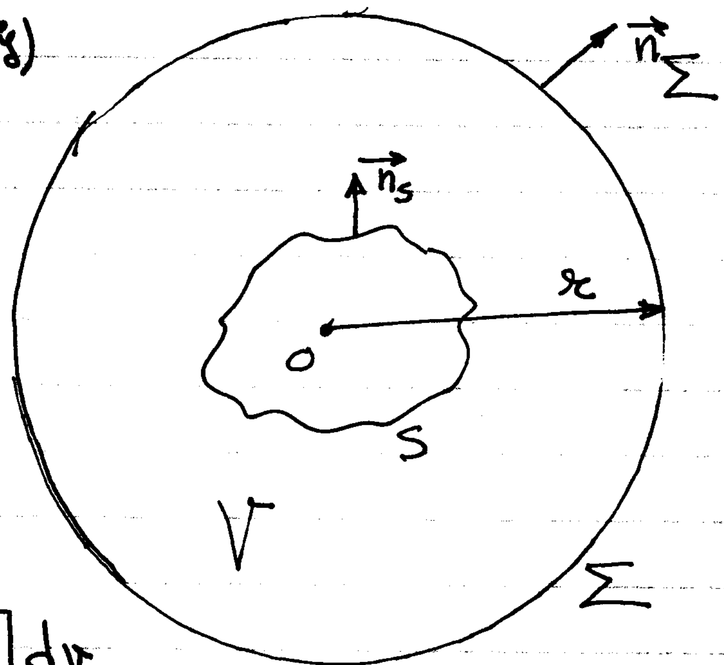
Green's function: g

$$(\nabla^2 + k^2)\, g(\vec{x}|\vec{y}) = -4\pi\,\delta(\vec{x} - \vec{y})$$

$$g = \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$$

Let S be a surface with unit normal $\vec{n}_s$ and let Σ be a large sphere of radius r, centred at the origin, $r = |\vec{x}|$, containing S with unit normal $\vec{n}_\Sigma$

$$(\nabla^2 + k^2)\hat{p} = i\omega\hat{q} + \nabla\cdot\vec{\hat{f}}$$
$$(\nabla^2 + k^2) g = -4\pi\delta(\vec{x} - \vec{y})$$



Then,

$$g(\nabla^2 + k^2)\tilde{p} - \hat{p}(\nabla^2 + k^2) g$$
$$= \nabla\cdot(g\nabla\hat{p} - \hat{p}\nabla g)$$

Hence, using the divergence theorem,

$$\int_V \left[g(\nabla^2 + k^2)\hat{p} - \tilde{p}(\nabla^2 + k^2) g\right] dv$$
$$= \int_\Sigma (g\nabla\hat{p} - \hat{p}\nabla g)\cdot\vec{n}_\Sigma \, d\Sigma - \int_S (g\nabla\tilde{p} - \tilde{p}\nabla g)\cdot\vec{n}_s \, ds$$

$$(\nabla^2 + k^2)\tilde{p} = i\omega\hat{q} + \nabla\cdot\vec{\hat{f}}$$

$$\int_V g(\nabla^2 + k^2)\tilde{p} \, dv = i\omega\int_V \hat{q} g \, dv + \int_V (\nabla\cdot\vec{\hat{f}}) g \, dv$$

$$\int_V \tilde{p}\,(\nabla^2+k^2)\,g\,dv = -4\pi\int \tilde{p}(\vec{x})\,\delta(\vec{x}-\vec{y})\,dv$$
$$= -4\pi\,\tilde{p}(\vec{y})$$

$$\tilde{p}(\vec{y}) = \frac{1}{4\pi}\Bigg\{ -i\omega\int_V \hat{q}(\vec{x})\,g(\vec{x}|\vec{y})\,dv_{\vec{x}} - \int_V (\nabla\cdot\vec{f})_{\vec{x}}\,g(\vec{x}|\vec{y})\,dv_{\vec{x}}$$
$$+\int_\Sigma (g\,\tilde{\nabla}p - \tilde{p}\,\nabla g)\cdot\vec{n}_\Sigma\,d\Sigma - \int_S (g\tilde{\nabla}p - \tilde{p}\nabla g)\cdot\vec{n}_s\,dS\Bigg\}$$

Exchange $\vec{x}$ and $\vec{y}$,

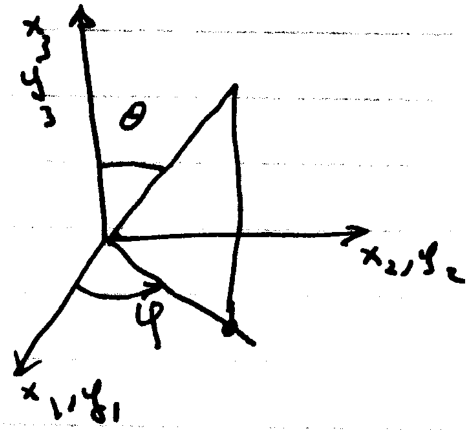
$$\hat{p}(\vec{x}) = \frac{1}{4\pi}\Bigg\{ -i\omega\int_V \hat{q}(\vec{y})\;g(\vec{x},\vec{y})\,dv_{\vec{y}} + \int_V (\nabla\cdot\vec{f})_{\vec{y}}\;g(\vec{x},\vec{y})\,dv_{\vec{y}}$$
$$+\int_\Sigma (g\,\hat{\nabla}p - \hat{p}\,\nabla\hat{g})\cdot\vec{n}_\Sigma\,d\Sigma - \int_S (g\nabla\tilde{p} - \tilde{p}\nabla g)\cdot\vec{n}_S\,dS$$

$$g = \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$$

$$\nabla g = \left(ik|\vec{x}-\vec{y}| - 1\right)\frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y})$$

Consider the surface integral on Σ.

$$d\Sigma = r^2 \sin\theta \, d\theta \, d\varphi$$



$$\nabla\hat{p}\cdot\vec{n}_\Sigma = \frac{\partial\hat{p}}{\partial r}$$

$$g\frac{\partial\hat{p}}{\partial r} = \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\frac{\partial\tilde{p}}{\partial r}$$

$$\hat{p}\,\nabla g\cdot\vec{n}_\Sigma = \tilde{p}\frac{\partial g}{\partial r}$$

As $r \to \infty$, $\nabla g \to \left(\frac{ik\, e^{ik|\vec{x}-\vec{y}|}}{r}\right)\frac{\vec{x}}{r}$

$$|\vec{x}-\vec{y}| = |\vec{x}| - \frac{\vec{y}\cdot\vec{x}}{|\vec{x}|} + \dots = r - \frac{\vec{y}\cdot\vec{x}}{r} + \dots$$

$$\nabla g \longrightarrow ikg\frac{\vec{x}}{r}$$

$$g\frac{\partial\tilde{p}}{\partial r} - \tilde{p}\frac{\partial g}{\partial r} = \frac{e^{ik|\vec{x}-\vec{y}|}}{r}\left(\frac{\partial\hat{p}}{\partial r} - ik\tilde{p}\right) + o\left(\frac{\hat{p}}{r^2}\right)$$

Sommerfeld radiation condition

$$\lim_{r\to\infty}\left[r\left(\frac{\partial\hat{p}}{\partial r} - ik\tilde{p}\right)\right] = 0, \quad \hat{P} \to 0$$

This means the integrand over Σ vanishes faster than $\frac{1}{r^2}$, therefore

$$\lim_{r \to \infty} \int_{\Sigma} (g \nabla \tilde{p} - \hat{p} \nabla g) \cdot \vec{n}_\Sigma \, d\Sigma = 0$$

$$\tilde{p}(\vec{x}) = \frac{-i\omega}{4\pi} \int_V \hat{q}(\vec{y}) \, g \, dV + \frac{1}{4\pi} \int_V \nabla \cdot \vec{\tilde{f}} \, g \, dV$$

$$- \frac{1}{4\pi} \int_S (g \nabla \tilde{p} - \tilde{p} \nabla g) \cdot \vec{n}_s \, dS$$

Note $$\frac{\partial \vec{v}}{\partial t} = -i\omega \vec{v} = -\frac{1}{\rho_0} \nabla p$$

$$\nabla p \cdot \vec{n}_s = i\omega v_n \rho_0$$

$$\tilde{p}(\vec{x}) = \frac{-i\omega}{4\pi} \int_V \hat{q}(\vec{y}) \, g \, dV + \frac{1}{4\pi} \int_V (\nabla \cdot \vec{\tilde{f}}) \, g \, dV$$

$$- \frac{i\omega \rho_0}{4\pi} \int_S v_g \, g + \frac{1}{4\pi} \int_S (\nabla g \cdot \vec{n}_s) \tilde{p} \, dS$$

For r large

$$g \longrightarrow \frac{e^{ikr}\, e^{-ik\vec{x}\cdot\vec{y}}}{r}$$

$$\int_V \nabla\cdot\vec{\hat{f}}\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}\, dV = +\frac{ik\vec{x}}{r}\int_V \vec{\hat{f}}\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}\, dV$$

$$\hat{p}(\vec{x}) = \frac{-i\omega}{4\pi}\frac{e^{ikr}}{r}\int_V \hat{q}(\vec{y})\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}\, dV$$

$$-\frac{ik}{4\pi}\frac{e^{ikr}\,\vec{x}\cdot}{r}\int_V \vec{\hat{f}}(\vec{y})\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}\, dV$$

$$-\frac{i\rho_0\omega}{4\pi}\frac{e^{ikr}}{r}\int_S V_n\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}\, dS$$

$$+\frac{ik}{4\pi}\frac{e^{ikr}}{r}\int_S \hat{p}(\vec{y})\; e^{-ik\frac{\vec{x}\cdot\vec{y}}{r}}(\vec{x}\cdot\vec{n}_s)\, dS$$